An Inquiry/Review into the Generalized Spectrum of Dimensions, D_q, and its Relevance to Research[i]

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Abstract

The generalized spectrum of dimensions of time series provides additional measures of a signal that single dimensional measures fail to offer: (1) when a single dimension is chosen it may not discriminate between experimental conditions when other choices might have succeeded. Running the spectrum provides a search of all possible dimensional measures. (2) parameters of the spectrum provide additional features such as asymptotes, inflections, etc. that may also reveal the effects of different experimental conditions. The use of multiple methods of measurement and the use of a comprehensive approach to experimental design and analysis is urged. Thus we examine an application of the generalized dimensions and graphic EEG presentation (Kulish, Sourin, & Sourina, 2006), and a comprehensive program of data analysis (Sprott, 2003).

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1. Introduction

Nonlinear measures of the dimensionality, D_q, of time series have usually used methods that yield a single dimension, such as the capacity dimension, q = 0), the information dimension, (q = 1), and the correlation dimension, (q = 2), to name the most familiar. D_2, developed by Grassberger & Procaccia (1983), is the most tractable and meaningful of these for experimental data. But D_q can be generalized for q as a continuous variable over a large range as a generalized spectrum of dimensions, also known as multifractals (Grassberger, 1983; Hentschel & Procaccia, 1983; Paladin & Vulpiani, 1987; Rényi, 1970) and well presented in (Abarbanel, 1996; Kantz & Schreiber, 1997; Ott, Sauer, & Yorke, 1994; Sprott, 2003). Despite the potential for improved information about the differences in experimental data that the D_q offers there are few applications that have been made to real data, but there are a few on the dynamics of heartbeat (Amaral *et al*., 2001), EEG (Kulish, Sourin, & Sourina, 2006), and tropical rainforest (Solé &

Manrubia, 1995). My purpose here is not to develop further the methodology, but (a) to try to emphasize its potential usefulness in order to get more researchers to explore that potential, and (b) to review overall data-analytic strategies in which this method would be imbedded. The strategy is mainly that of Sprott (2003) whose comprehensive program is outstanding but, unfortunately, seldom used.

I start with an examination of the Kulish *et al*. paper on EEG and language. They state their goal as "developing new methods of processing data recorded by well-established techniques [that] may prove useful while deeper penetrating into the mystery of human consciousness." Since the brain is non-linear and FFT power spectral analyses are linear, non-linear measures should prove useful. Their use of the D_q is valuable in that regard. I also include a presentation of their graphic methods of evaluating EEG because multiple methods of evaluating signal differences complement each other. I comment on ways of increasing the cross-relevance of their dimensional and graphic methods in my 'wish list', a discussion of research design and analysis strategies.

Note that it is reasonable to consider that the linear spectra of frequency decomposition of time signals is indeed useful, but that multiple analytic methods might supplement with additional information that can be extracted from the signal(s) including nonlinear and graphic measures. For example, in R. Abraham & C. Shaw (*Dynamics: The Geometry of Behavior*), they have a figure comparing the representations of various attractors with three images each: the portrait, the time series, and the frequency spectrum (Part 2, Fig. 4.5.7; repeated in Abraham, Abraham, & Shaw, Fig. II-49. These books also show the relationship to characteristic exponents (Liapunov exponents), but not to fractal dimension, the subject of this paper). As Abarbanel (1996, p. 69) puts it, "Since chaotic motion produces continuous broadband Fourier spectra, we clearly have to replace narrowband Fourier signatures with other characteristics of the system for purposes of identification and classification." Then he mentions fractal dimensions and Lyapunov exponents as the two main candidates as classifiers.

2. Mathematical Analysis

Kulish *et al*. begin with the Rényi entropy measure, which seemed curious at first since it does not use sequential properties of the data, but rather is the probability distribution function of the time signal (i.e., the EEG voltages). Buat from it they developed the generalized fractal dimension, D_q, the convolution over the probability distributions of order q summed over the bins of the EEG voltages and the size of the hypersphere, δV (which is equivalent to r or ε of the usual formulations of D since there is a measured dimension, voltage, associated with r). This is their equation 6 (Sprott, §13.2.2, eqn. 13.14, p. 338; Kant & Schrieber, §11.3.1 eqn. 11.12, p. 187; Ott, Sauer, Yorke, §2.2, eqn. 2.2, p. 15; Abarbanel, §5.2, eqn. 5.12, p. 73, Renyi, 1970; Grassberger, 1983; Hentschel & Procaccia,1983; Paladin & Vulpiani, 1987.) The resulting D_q as a function of q, is sometimes called the fractal spectrum, also, the generalized Renyi entropy, generalized fractal dimensions, generalized dimensions, the spectrum of fractional dimensions, multifractal spectrum of dimensions, and multifractal.

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Further, the generalized fractal dimensions of a given time series with the known probability distribution are defined as

$$D_q = \lim_{\delta V \to 0} \frac{1}{q-1} \frac{\log \sum_{i=1}^{N} p_i^q}{\log \delta V}, \tag{6}$$

Of the four books I have mentioned, any one of them presents a more than adequate presentation of the mathematics involved, but each has strengths that warrant owning all of these books. Abarbanel's (1996) is excellent for data visualization, especially on explaining the box counting method with figures for the Hénon attractor (reproduced in Abraham, 1997). Kantz & Scheiber's (1997) is especially good in pointing out (a) the pitfalls when applying the techniques (§6.4), (b) when automated algorithmic methods work well, and (c) when the art of looking at the graphics of intermediate steps is needed in an analysis. They also caution about the of loss of information in trying to characterize a dynamic by reduction to single numbers—e.g., p. 38 on visual inspection of data; p. 72 on independence from measurement and analysis parameters. They are also good on stressing the use of multiple approaches to data analyses. They have an appendix containing some of their numerical routines in both Fortran and C. Ott *et al*. (1994) have a unique organization by with general explanatory chapters each of which is followed by many reprints of classic original papers, a must-have reference work from that perspective. Sprott (2003) is excellent for (a) clear explanations, (b) completeness, (c) appendices including a huge list with summaries of known attractors, (d) historical vignettes on many the authors in the history of dynamics, and (e) a web site that helps with exercises and which keeps the book constantly updated.

3. Experimental technique and signal processing.

I was surprised to see EEG of 10-15 mV p/p; sure beats the days when I did EEG from indwelling electrodes (theirs was scalp), which might explain why they could be so casual with respect to shielding and isolation of the subjects, and with respect to grounding issues (not enough information but I assume they had no problem with ground loops). Mine (indwelling, cat) were more in the range reported by Babloyantz (1988/1990, see Fig. 2 showing typical ranges of EEG voltages) At any rate, their signals looked pretty clean. I was curious about their putting cotton balls behind the ears. Could that have been for EOG rather than ECG? A misprint? Statistical processing and "the significance test was performed". Which tests and on what?

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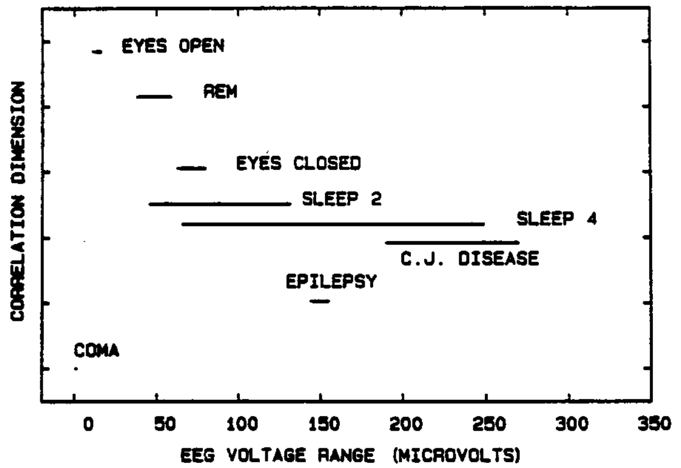

Fig. 2. Dimension-amplitude plot. The variations of EEG voltage observed at the optimal sampling frequencies (see text) and in successive stretches of 1 s are represented for each type of EEG behavior. The synchronization between neurons which occur in pathologies is reflected by high amplitudes and low dimensions

Their behavioral task consisted of asking the subject a question to which they had to respond 'yes' or 'no'. Then they performed the D_q on the EEG to each of these responses. The precise timing of the analysis period and the response was not given. Presumably, these EEG samples were taken during the utterance, although the 5 sec EEG sample is considerably longer than the utterance of the answer would take.

4. Fractal spectra of EEGs.

They state that the Fourier transform does not yield information about amplitude, but it does provide amplitude information in a relative sense at least (power per frequency band; when the signal is in arbitrary units), and can be true in the absolute sense of the distribution of $V^2/\delta f$ as a function of f if calibrations are made which reveal the transfer function of the measuring system (*sans* electrode/brain interface in our work (Abraham, F. D. Brown, D., & Gardiner, M. (1968). Hard to believe it had never been done before. We did it with a train of rectangular pulses whose convolution yielded the whole spectrum with a single signal.) While they accuse the Fourier of also not yielding fractionality information, it is just as true that while the fractal dimension is sensitive to frequency information, it does not yield any frequency information directly. Is it hard to believe that if the D_q discriminated between the responses, the Fourier would also not yield a difference if performed with appropriate parametric choices? These are minor points and not very consequential considering the important methods that they are developing.

While the EEG voltages may have been large compared to EEGs reported earlier, the opposite is true for the fractal dimensions. After Grassberger & Procaccia first reported their algorithms for measuring D_2 in 1983, there were a spate of articles reporting D_2 and attractor reconstructions for EEG, mostly with D_2s of 4-6 compared to the <2 values reported in Kulish (Başar & bullock, 1989; Başar, 1990, see Table 2, Introduction) 2. Why might this be? One can only speculate, since there were not many details of the analysis procedures given. But perhaps we can get a clue from Kantz & Scheriber (1997), §6.3, where they point out that calculating the correlation sum and dimension involve attractor reconstruction, which involves the delay imbedding procedure (where calculating the delay τ is critical); "A value which "yields a convincing phase portrait should do for the correlation sum as well." Also, results should be "invariant under reasonable changes to the embedding procedure." (p. 73.)

> "Once the embedding vectors are reconstructed, the estimation of dimension is done in two steps," determining the correlation sum, $C(m,\varepsilon)$, where m is the embedding dimension, ε (= r of others), the radius of the covering hypersphere) and then "inspect $C(m, \varepsilon)$ for signatures of self-similarity. If these signatures are convincing enough, we can compute a value for the dimension. Both steps require some care in order to avoid wrong or misleading results." (p. 73.)

> "The straightforward estimator, Eqn. (6.1), turns out to be biased toward too small dimensions when the pairs entering the sum are not statistically independent. For time series data with nonzero autocorrelations, independence cannot be assumed. . . The most important temporal correlations are caused by the fact that data close in time are also close in space" [For the EEG in Kulish *et al*., the state space is based on voltage]. "It is more than likely that the majority of dimension estimates published for field measurements are seriously too low because they mistake temporal coherence for geometrical structure." (Kantz & Schreiber, p. 73; demonstrated by Theiler, 1986).

The solution involves decimating the time series to eliminate the correlations. Kantz & Schreiber's next section, 6.4, shows how $C(\varepsilon)$ as a function of ε and $d(\varepsilon)$ as a function of ε with parameter m is examined for linear portions (those signatures of self-similarity). The finite nature of the attractor and differing densities of points at different portions of the attractor affect the statistical reliability.

Sprott summarizes some 11 steps or procedures for the analysis of time series data (Sprott, §13.8; summarized here in §6 below). Many of these procedures are not mentioned or utilized or are incompletely described in the Kulish paper, so no estimate can be made if their procedures satisfy these precautions. Nonetheless, the use of the apectrum of generalized dimensions, D_q, is an extremely important contribution.

One might remember that in reconstruction work, a usual criterion for selecting a lag, τ, is to use the delay required for the autocorrelation to decay from 1 to 0.

Their main results for one subject are in their Fig.3, and averaged over all subjects, substantially identical, are in their Fig. 4. They certainly are well behaved. The curves for "yes" answers are higher than for "no" answers, as is D_0 (Hausdorff-Besicovitch version). In addition, the range of D's from $D_{-\infty}$ to $D_{+\infty}$ was also greater for "yes" than "no" answers. Following the latter result, they state, "Hence, the 'YES' spectra are *on the average* more fractal than the 'NO' spectra. This implies that the brain is *on the average* more active while answering 'YES' question. In addition, it is evident from Fig. 4 that *on the average*, more unexpected values are present in the 'YES' signal, whereas the 'NO' signal is *on the average more* predictable and uniform."

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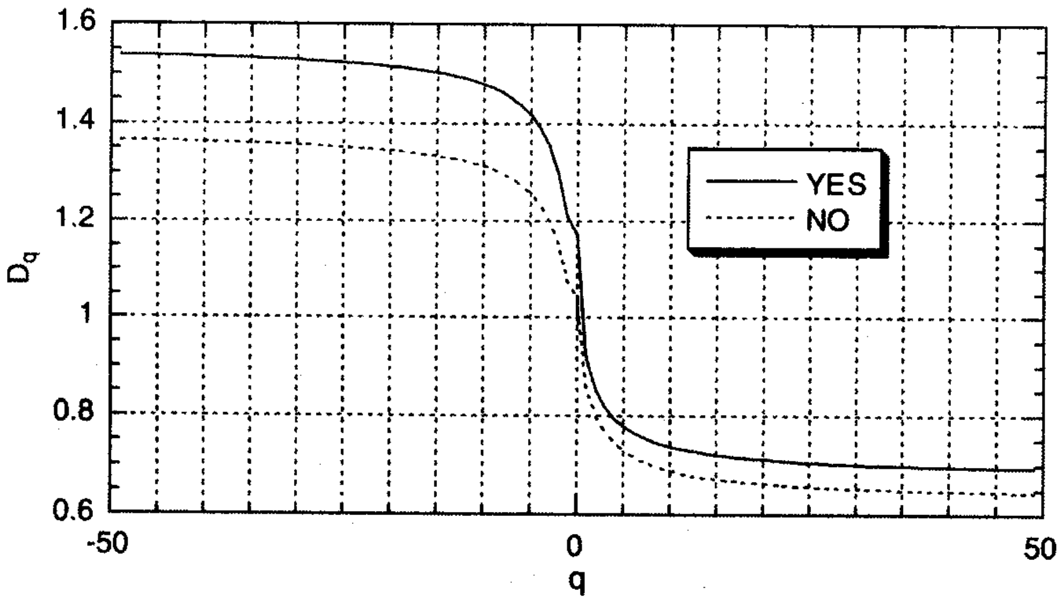


Fig. 4. The sample averaged fractal spectra of EEGs.

The conclusion that the spectra are greater for the “yes” answers certainly seems safe, even without error boundaries for the curves. The conclusion that the brain is more active is not necessarily wrong, but requires some conjecturing to have any meaning. It is consistent with the typical results, e.g., the Figure 2 of Babloyantz (1989/1990; see supra) which shows D_2 as a function of voltage for disease (epilepsy, Creutzfeld-Jakob), sleep (stages 2, 4, and REM), and eyes closed and open, with D_2s increasing from about 2 to 10 (as voltage ranges decrease). A clue to interpreting such a result is given by Babloyantz: “The synchronization between neurons which occur in pathologies is reflected by high amplitudes and low dimensions.” (see caption of her figure.) That is, it is not necessarily more or less activity, but the coherence of neuronal activity that is being broken down into subpopulations of neurons and involved in more information processing tasks occurring at any time (also, see Abraham, 1997; Abraham *et al*., 1973 on coherence and brain transactions). Why would “yes” involve such an increase in information processing in the brain. While it is impossible to tell from this experiment, or to conjecture with any confidence (i.e., regarding the involvement of greater emotion, less security, more information to evaluate cognitively), it would be of value to use signal-detection theoretical methodology and interrogation of the research participants to try to understand emotional and cognitive strategies.

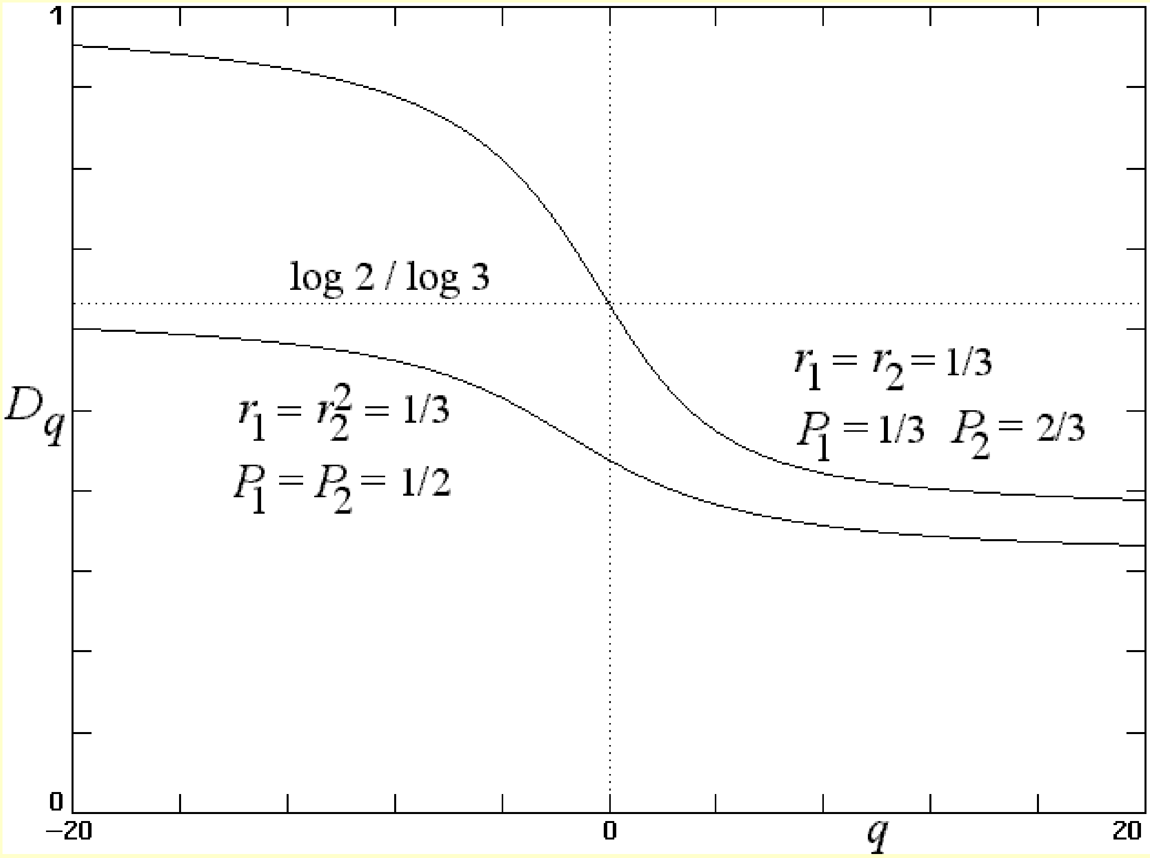
What do we make of the fact that the range of D_q is greater for “yes” than “no”? Probably not much. The increase was due to the negative end of the spectra (where $q < 0$) which is more sensitive to the less dense portions of the attractor, which is where the more extreme voltages are. Mutlifractals are designed to overcome problems of committing to a particular D due to the variability in the density (probability of occupancy) of the attractor. Sprott (2003, p. 338): “The limit [of D_q as] q approaches $+\infty$ gives the local dimension in the most densely populated region of the attractor, and the limit q approaches $-\infty$ gives the local dimension in the least densely populated region. The former can be calculated more accurately than the latter because there are usually many more points in the denser region (which is how it got to be dense!)."

The "yes" signal has more unexpected values; the "no" more predictable. Do we need D_q to tell us that? Not only is it obvious from the time signal (EEG), but wouldn't a simple frequency (probability) distribution and/or its statistical moments tell us the same? Doesn't this result tell us as much about D than D does about predictability? Of course the big question is: does signal predictability reflect cognitive predictability? This is an interesting question that should be amenable to further experimentation.

Kulish *et al*. comment that the information dimension, D_1, higher for "yes" than "no", is unexpected since they should have "equal informational content". However it is not necessarily the case that the cognitive and emotional requirements (nor the brain processes required) for the two answers should be the same, and indeed, the fact that the whole spectrum (not just D_1), is different for the two answers would seem to indicate that to be the case. But with the respect to the EEG signal, D_1, which equals the limit of the Shannon entropy [Sprott: "the amount of information required to specify the state of the system"] divided by log δV as δV approaches 0, which as Sprott observes, "describes how fast the information needed to specify a point on the attractor increases as r [δV in this paper] decreases." (Sprott, 203, p. 339.)

Kulish *et al.* make a further conjecture that "this result [unequal D_1s] can be viewed as an indirect proof of the fact that the operation of the brain is fuzzy, that is there is an overlap between the set in question ("YES") and its complement ("NO")". Very indirect!! Fuzzy?! What can this mean? While yes and no may be complementary as far as the language goes, the expectation that completely different brain processes might support them is an unlikely hypothesis. Suppose the brain was divided into two sets of neurons that only talked to neurons in their own set. Might we not have achieved the same result? And why, as they claim, should the D_1s add up to 2 bits as they claim? [$D_{1,yes} = 0.921$ bits; $D_{1,no} = 0.853$ bits; $\Sigma = 1.774$ bits, whereas, if independent the sum should be 2 bits according to Kulish *et al*. I am a bit confused as I didn't think the limits defining D, even the information dimension D_1 had units or dimension of their own, so this argument escapes me; but I am on unsure ground on this.] Babloyantz "eyes closed" and eyes open" conditions each had D_2s (which must be very close to D_1 as the D_q curves all have the same shape, and adjacent D_qs are very similar) >2. There likely is an overlap in the processing of information in making the binary decision, but it is hard for me to see how it follows from the non-additivity of D_1s of the EEGs for the two responses. In a final observation about the fractal results, Kulish *et al*. point out that the D_q curves are nearly identical to that for the logistic equation (but they don't specify the parameter of the logistic).

It may be relevant that Sprott shows nearly identical curves for the logistic equation (he specifies the logistic equation's constant for different multifractals and approximates the logistic with an asymmetrical Cantor set, (§13.4.2, Fig. 13.11, pp. 342-243) and for a pair of asymmetric Cantor sets (§13.3.1, Fig. 13.8, pp.340-341.) [One might note, *a la* the Kulish argument, that these curves do share the same brain, that is the same equation except for control parameters, with the sum of D_1s in the neighborhood of 1.]



Sprott: Figure 13.8, D_q for two different asymmetric Cantor sets. (Thanks to Sprott for sending an electronic copy of this figure.)

The fact that two identical deterministic difference equations deferring only in the choice of their parameters yield similar curves whose difference can be specified with but a few parameters (amplitude and range of D_q, D_0, slope of the tangent at the inflection point at D_0) does not imply that two real-world processes, such as the cognitive ability to answer ‘yes’ or ‘no’ to questions, are generated by either similar or different processes. Nor does it tell you whether even the processes are deterministic or stochastic (where the difference equations for the latter would have to have a probabilistic component). It does mean that the dimensional magnitude of one is greater than the other. If deterministic equations represent similar processes, in what way are they fuzzy other than that they shared some components of a process but not others? Because an attractor is chaotic or possesses fractal dimension and can be characterized with Shannon entropy neither makes nor disproves the probabilistic case. For those who wish to involve stochastic resonance, specific theories would have to built on the brain/cognitive processes involved in which noise contributes to hill climbing from the basin of one attractor into that of another, and I suspect such attempts would likely to be rather simplifications or the several cognitive/emotional processes involved.

5. Visualization

This is a really exciting methodology, and its potential use in psychophysical and psychological research and medical diagnosis and therapy is vast. I am not current on what other similar methodologies might be around, but this one looks sophisticated to me.

In brief, the magnitude of EEG is evaluated at each point on the skull, which can be tracked in real time with a visualization by means of 'blobbies', partial spherical forms on a mannequin head, by means of a graphics computer technology by an award winning distinguished team of biomedical engineers.

The "blobby" is defined by equation 12, p. 298:

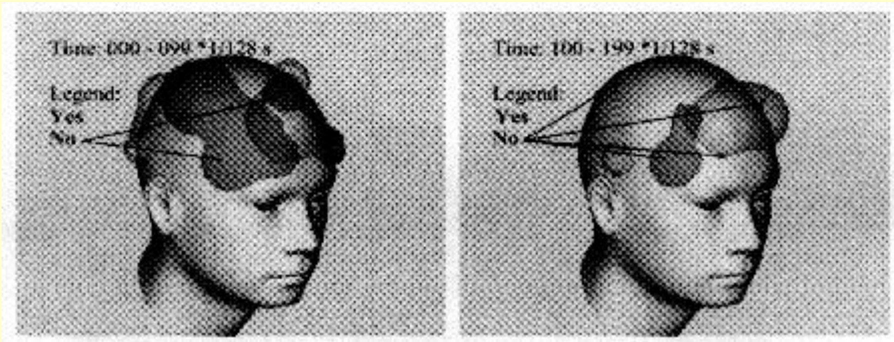
$$f(x, y, z, t) = \sum_{i}^{24} a_i \mathrm{e}^{-r_i \cdot b_i(t)} - g \geqslant 0,$$

$$r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2},$$

"where *a* is a scale factor, *b* is an exponent scale factor and *g* is a threshold value."

The index *i* is over the EEG channels. *x,y,z* were coordinates in a 3D Cartesian space around each electrode position, and I presume when unindexed, some sort of average. The distance of the electrode to the reference position being r_i. The voltages enter via *b*, and are thus the factor that affects the size of the blobby. More than one blobby can be shown at a time to compare different aspects of an experiment and different loci of activity and their magnitude. The threshold g requires a level of EEG activity before a blobby is visualized for any given electrode location. A blobby is calculated for a given time position, but this can be advanced for real time movies or snapshots for particular time windows, advancing through the 5 sec of the response window.

Here is an example of the first and second .77 sec epochs during an experiment, showing blobbies superimposed on the head for 'yes' and 'no' responses, the color difference surviving as shades of gray here and in the article. It nicely shows the reduced area of cortical activation (it became more so over the ensuing 6 epochs). (Notice the original 256 Hz digitization rates are decimated in half.) Interactive computer windows and palettes allow real time changing of both viewing and computational parameters. Boolian operations allow additional experimental evaluations.



From Figure 6, p. 300.

They conclude that EEG activity is greater for ‘yes’ responses, but that “no” responses require less mental activity and is more stressful assumed from its lesser cortical involvement but greater persistence in the visual cortex.

Despite the sophistication of the visualization methods, I think there are other measurements in addition to EEG amplitude including power spectra and co-spectra, and especially EEG coherence measures, and most especially, some of the parameters of the generalized fractal spectra that could be fed into the visualization procedures that might help with more subtle cognitive interpretations. Additionally, I would love to see more complete results for the experiment, and think that any conclusions based on such a brief methodological report would be very exciting when used in a more extensive and rigorous experimental context.

6. My wish list.

a) When unusual statistical and geometrical representations are used, whether D_q or blobbies, for which the probability distribution functions and their higher moments (variance, skewness, kurtosis) may not be estimated, it is often helpful to evaluate their reliability and significance with Monte Carlo methods. They would constitute a welcome addition to the present study. (Abraham *et al*., 1973; Abraham, 1997).

b) It is nice to have calibration of EEG signals and their derivative measures demonstrated (Abraham, Brown, & Gardiner, 1968).

c) Utilizing as many of Sprott’s 11 steps or procedures for the analysis of time series data as possible (quoted or précis from Sprott, 2003 §13,8, pp. 348-349:

i. Make sure the data are free of errors.
ii. Test for stationarity.
iii. Explore a variety of plotting the data.
iv. Determine the correlation time or minimum of the autocorrelation or mutual information to optimize the sampling rate.
v. Check the autocorrelation function or fourier spectra for periodicities.
vi. Make a time-space plot to be sure there are enough data.
vii. Use false nearest neighbor or saturation to establish the embedding dimension in the determination of D_2.

viii. If the embedding dimension is low, determine D_2.
ix. If there is a low dimensional attractor, compute Lyapunov exponents, entropy, and growth rate of unpredictability. If high, remove noise.
x. If there is chaos, use surrogate methods for the above tests.
xi. If there is low dimensional chaos, construct equations and make short-term predictions. If high dimensional chaos (more common), “some predictability [maybe] is possible, and whose power spectrum and probability distribution allow comparison with theoretical models.”

d) Considering the subtlety, multiplicity, and complexity of transactions within the brain, and the largely unknown nature of them for the subtleties of cognitive processes, it is not surprising that the best of our analytic techniques are frustratingly inadequate at revealing those subtleties. I am particularly thinking of the relationships which may be taking place between different areas of cortical (and subcortical when indwelling electrodes permit), so when blobbies indicated more than one active area, it could prove of interest to see if the EEG in those areas would show some covariance/coherence (Abraham, 1997; Abraham *et al*., 1973 for power spectra, but could be done with with $b_{i,j}$ here). And in fact, all pair-wise sets of electrode results could be funneled into a discriminant or other canonical correlational analysis. And especially, these analyses could be fed into another round of blobby movies.

e) When this paper segued to the visualization I thought they were going to visualize D_q, or at least D_0, D_1, or D_2, or defining parameters of Dq, so I put that on my wish list also. If you put all these together, blobbies for D, voltage, and paired covariance of voltage, along with Monte Carlo and surrogate methods, and the rest of Sprott’s steps, then the methods so cleverly developed here would more definitely realize their potential.

f) And finally, some refinement of reporting of experimental procedures, such as the temporal indication of stimuli and response with the EEG recording would prove a benefit. Someone taking advantage of all of these would then be on the threshold of the evolution of an exciting experimental program.

The authors are to be congratulated and thanked for developing these tools for the rest of us to use. I am certainly appreciative of the opportunity to learn as much as possible from this exercise of trying to understand their work, despite my own limitations. I submit it here in hopes that others will become interested in this work and that of the other authors cited.

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