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A BEGINNER'S[1] GUIDE TO THE NATURE AND POTENTIALITIES OF DYNAMICAL AND NETWORK THEORY PART II: A VERY VERY BRIEF COMPARISON OF DISCRETE[2] NETWORKS TO CONTINUOUS DYNAMICAL SYSTEMS

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If you want to inspire confidence, give plenty of statistics — it does not matter that they should be accurate, or even intelligible, so long as there is enough of them.
Charles Lutwidge Dodgson, *Three Months in a Curatorship*, 1886.

As a model of a complex system becomes more complete, it becomes less understandable.
John M. Dutton & William H. Starbuck,
Computer Simulation of Human Behavior, 1971

ABSTRACT AND INTRODUCTION

Systems are collections of things that interact over time. I could give many examples, but then, so could you, since that includes just about everything in the universes. Many theoretical schemes have been used to model and measure system's behaviors. I review and compare the rudiments of just a few here, namely dynamical systems (summarized from Part I), Boolean networks (Kauffman), especially cellular automata, but saving scale-free networks (Barabási) for Part III.

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[1] The author is the beginner, who invites others to share this introductory exploration. I minimize citations to those that are my principal sources and serve as a gateway to the vast literature on these subjects.

[2] 'Discrete' here refers to variables that can assume one state at a time from a discrete, usually relatively small set of distinct states.

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1. Review of Selected Features of Continuous Dynamical Systems

Basic representations of dynamical systems are (1) system diagrams; (2) when possible, by ordinary differential equations or difference equations for continuous or discrete time respectively, (3) time series of the variables involved, generated by integration of the system of equations, (4) Geometric portraits summarizing essential features of patterns of a system's behavior, which are of three main types, vectorfields, attractor-basin portraits displaying trajectories of variables and their critical features in state space, and bifurcation portraits, which are comprised of important bifurcational features as a function of system parameters , and (5) various statistics of their nature and behavior.

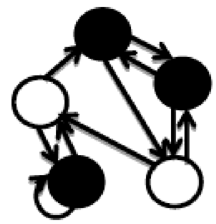
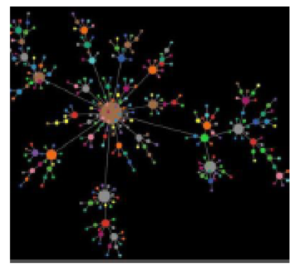
		
Dynamical System Double Pendulum Chaotic Trajectory by Long Exposure of LED on the Pendulum[4]	Dynamical Network Boolean 5,2 Network Binary Nodes Can Change After Mitchell, 2009	A Scale Free Network Nodes, Connections, and Number of Nodes Can Change[5]

Figure 1. Three Systems (footnotes[4,5]).

Figure 2 shows the first four of these representations. One does not usually show a system diagram for dynamical systems such as the Rössler system, but we deployed it here to facilitate comparison of these systems to networks. The parameters used here are the ones most typically used. Rössler's original values are also still in frequent use: a = 0.2, b = 0.2, c = 5.7.

Two of the most common of statistical properties used to characterize the behavior of chaotic attractors generated by systems such as Rössler's are the fractal dimension and the Lyapunov characteristic exponent, or rather a vector of them, the Lyapunov spectrum. The fractal dimension (Mandelbrot, 1983; Abraham, 1997, 2008) is a measure of the extent to which the attractor fills the Cartesian space in which it is embedded. For example, the Hausdorff (also called the capacity or box-counting) dimension, for the attractor shown, equals 2.01 ± 0.01which means it hardly takes up much space in its 3D home.[6] Of greater interest is the Lyapunov spectrum, λ_i, i = 1, 2, …n, n being the number of dimensions of the

4 Retrieved from Wikipedia, *Double pendulum*, 4/2/2011 7:45 AM, Includes applet simulation by Peter Lynch.

5 Made by running Lada Adamic's modification of a NetLogo library's preferential-attachment program, (p=1). http://ladamic.com/netlearn/NetLogo501/RAndPrefAttachment.html It also was in her Coursera course.

6 Sprott, 2003, p. 330; also http://en.wikipedia.org/wiki/List_of_fractals_by_Hausdorff_dimension

embedding space (one axis per variable). For the Rössler attractor, the global exponents (averaged over the whole attractor), $\lambda_1 > 0$, $\lambda_2 = 0$, $\lambda_3 < 0$. The λ_1 measures the rate of divergence and λ_3 measures the rate of convergence to the attractive manifold (globally) or from any two points in a basin in state space (locally). The absolute value of $|\lambda_3|$ is much greater than that of $|\lambda_1|$ which keeps the manifold thin and occupying so little of the embedding space (Sprott, 2003, pp. 118-120[7]).

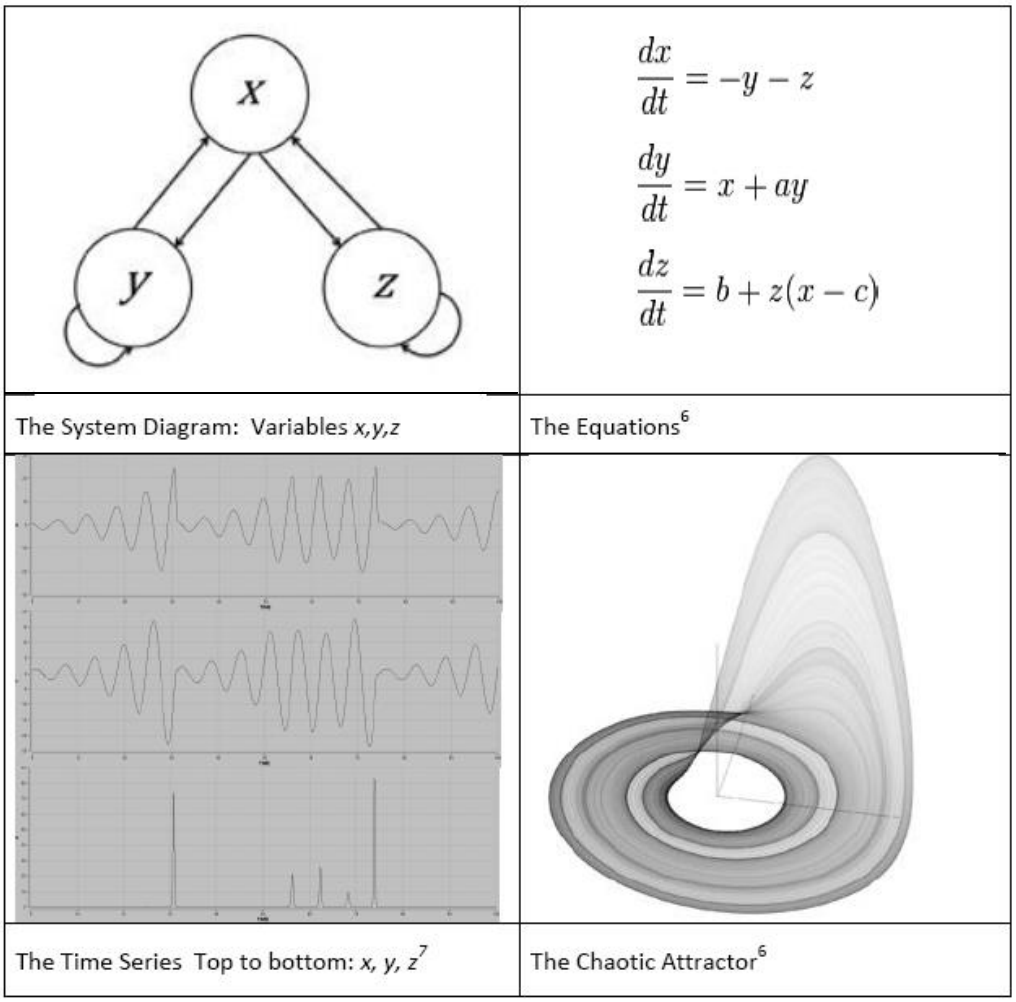


Figure 2. The Rössler System ($a = 0.1$ $b = 0.1$ $c = 14$).[8]

[7] Also at: sprott.physics.wisc.edu/chaos/com

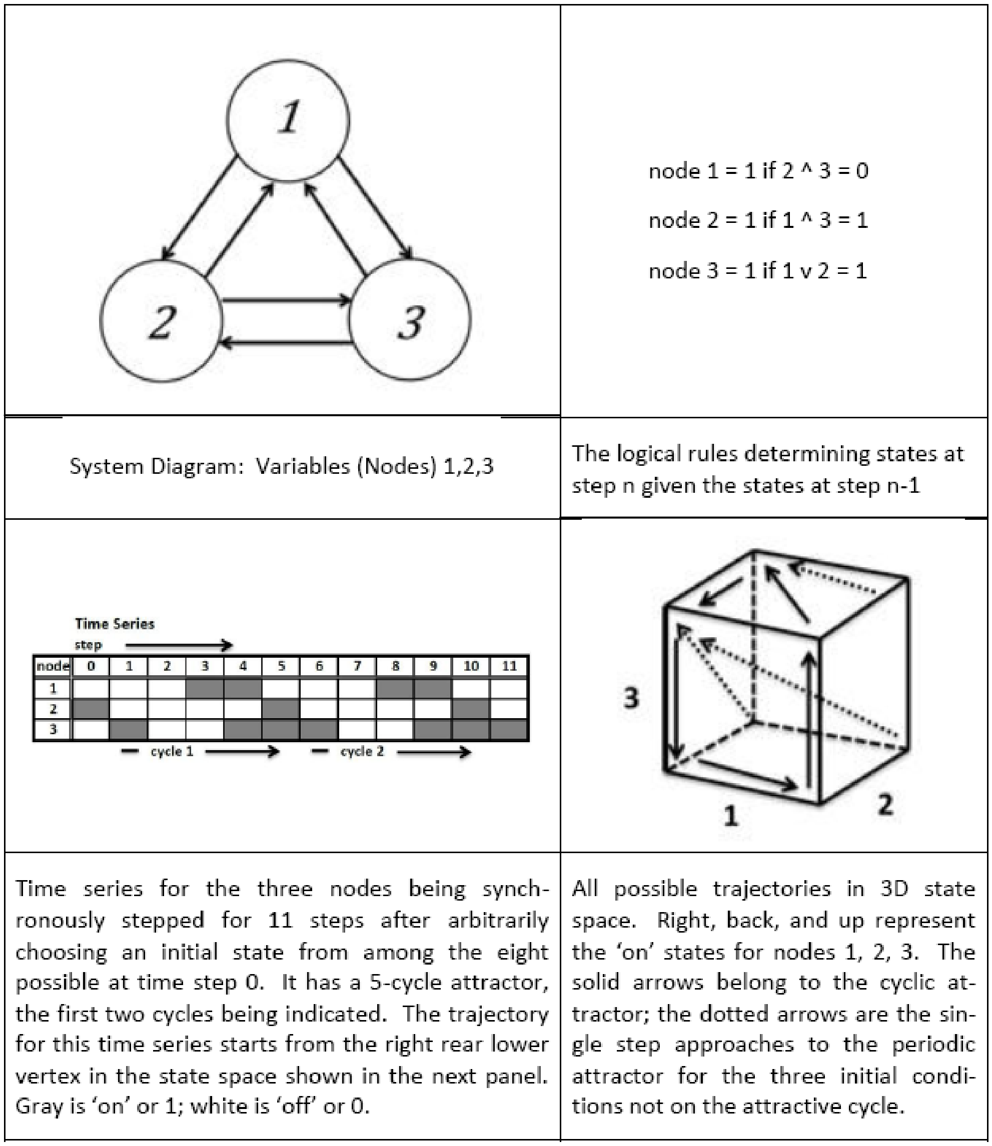
[8] Figure 2: Equations and Attractor from http://en.wikipedia.org/wiki/R%C3%B6ssler_attractor Retrieved 5 April 2011 (made by Wofl, 2005) . Diagram by me; time series Created by the author using Berkeley Madonna 8.3.18 by Robert I. Macey & George F. Oster

2. Review of Elementary Features of Boolean Networks[9]

Boolean networks are discrete networks where the system variables, usually called nodes, cells, or agents, are binary variables, that is can assume one of two values, usually represented by 0 and 1, but can also be expressed as on-off, black-white, 0 or 3 volts, dominant or recessive, true or false, etc. Connections or links can represent a variety of relationships between nodes that influence their change in state from step to step in the iteration process. A collection of nodes and their connections constitute a network or graph. They are a special case of cellular automata which were introduced by Ulam and von Neumann (von Neumann, 1948/1951, 1949, 1966). These were extensively developed by Kauffman (1969, 1993, 1995) and his colleagues to explore genetic, phylogenetic, ontogenetic, and epigenetic issues.

Basic representations of Boolean networks, similar to those of dynamical systems, include (1) system diagrams; (2) logical rules for changing the states of nodes from one step in time or sequence to the next instead of the equations used in dynamical systems—in the usual and simplest case, all nodes are stepped synchronously; (3) time series; (4) attractor-basin portraits (Hanson & Crutchfield, 1988) and bifurcation diagrams, and (5) various statistics of the nature and behavior of the system. (Figure 3 below illustrates the first four of these representations.)

[9] These are a species of cellular automata which were introduced by von Neumann (1966).



System Diagram: Variables (Nodes) 1,2,3

The logical rules determining states at step n given the states at step n-1

Time series for the three nodes being synchronously stepped for 11 steps after arbitrarily choosing an initial state from among the eight possible at time step 0. It has a 5-cycle attractor, the first two cycles being indicated. The trajectory for this time series starts from the right rear lower vertex in the state space shown in the next panel. Gray is 'on' or 1; white is 'off' or 0.

All possible trajectories in 3D state space. Right, back, and up represent the 'on' states for nodes 1, 2, 3. The solid arrows belong to the cyclic attractor; the dotted arrows are the single step approaches to the periodic attractor for the three initial conditions not on the attractive cycle.

Figure 3. Boolean 3,2 Network.

Kauffman's Random Boolean *NK* networks initially considered that *K*, the number of inputs or in-degree for all *N* nodes to be the same. This is a major limitation for modelling real systems, but a good starting place for studying general properties of system dynamics. When testing models, he chose (1) the input connections, (2) the logical rules of the nature of the switching from one state to the next for each node as a function of the state of the nodes from whence their inputs came, and (3) the initial set of states for the nodes.

The state space for such a network is *N*-dimensional, and the number of possible states in the state space is 2^N, or 8 for the three node case, represented as the 8 vertices of the cube showing the state space for the example in figure 3. This is not only finite, but dictates that the trajectory must become a fixed point or cyclic attractor within 2^N steps. Thus, the trajectory in figure 3 shows the 5 possible starting states within the attractive cycle which is shown by the solid arrows, and the 3 remaining starting points, each of which take one step to enter the cyclic attractor (dotted arrows). Note that the logical rules for a transition step could be also stated as a truth table.

A one-dimensional cellular automaton is the ring lattice, where every node in the ring has the same truth table (see figure 4 and table 1 below). Here is the truth table for a ring using r = 1 (r is the radius of the neighborhood of each node which is comprised of itself and r nodes to each side of it, the nodes having directed inputs to it). In this example the change of any node, x, from step n to step n+1 depends on its own state as well as the neighbors within the radius. The number of entries or rules in the truth table is $2^{(2r+1)}$, in this case, 8. One can make the outcomes after the step according to the logic rules, the truth table, established arbitrarily, randomly, or by some rational procedure, such as a genetic algorithm. Later we will see how changing the proportion of 0's and 1's in the rule table, λ, for running simulations of models to determine dynamical properties of cellular automata. For the ring lattice of the table and Figure 4, λ = 0.5. The logic table is applied to every node for this model; the possibilities for variations of such models can be appreciated to be vast.

Table 1.

Truth Table for Ring Lattice with r=1			
(node x, step n)			
(x-1,n)	(x,n)	(x+1,n)	(x,n+1)
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	0
0	0	0	0

For the ring lattice shown (figure 4) at step n obeying the logical (truth) table, going from step n to step n+1 would render node x=2 as still filled (filled=1; rule from row 6), while node x=9 would turn white (white=0; rule from row 2). Try it on some other nodes.

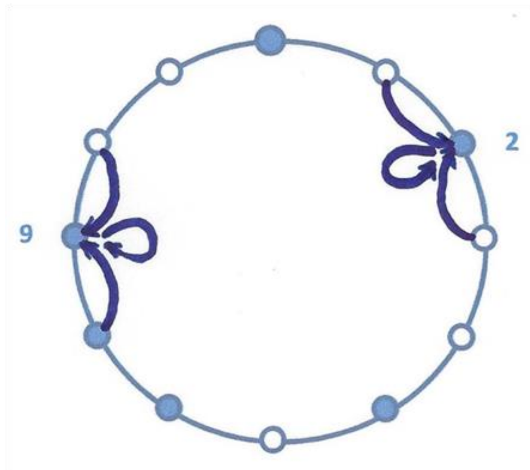


Figure 4. A ring lattice (above)

Figure 5. A Moore neighborhood which is comprised of a node and its eight neighbors. It would have a truth table of $2^9 = 512$ logical rules.

3. Chaos, Bifurcations, and Stability: Similarities of Continuous and Discrete Systems

3.1. Chaos

So, what is chaos? How does one get it? For small *N* and stationary conditions (the logical rules do not change), chaos will not show its beautiful face. But wait! Even when non-stationarity occurs, one simply is switching from one finite cycle to another, and there are only a finite number of rule changes possible, thus chaos is a transient complexity, however prolonged, within a cyclic attractor of a period so long that has exhausted your patience.

Mitchell makes the similar point that periodic attractors usually are shorter than the maximum possible with discrete networks which implies that longer cycles are possible within which chaotic complexity must occur (Mitchell et al. (1993, p. 4). So does Kauffman:

"But if the state cycle is too vast, the system will behave in a manner that is essentially unpredictable." (Kauffman 1995, loc. 1056[10].)

That sounds like chaos, at least within the limited meaning of unpredictability for chaos in continuous dynamical systems. We contend the some complexity within a periodicity holds true for continuous dynamical systems as in Boolean systems. It would seem that all chaotic attractors are transitional to fixed point or periodic attractors.[11] What we mean by this is that some attractors, if examined within but a portion of their trajectory to the end of a cycle, may exhibit complexity. We can describe some of this complexity as due to a mixture of divergence and convergence summarized by their Lyapunov exponents, without being concerned whether we have exploited time sufficiently to exhibit the cyclic nature of the system. Such concessions seem reasonable to model real systems in the time and space domains that interest us. This aspect is a similarity between continuous and discrete systems.

In examining the implications of Boolean networks in the interests of parsimony for exploring autocatalytic biochemical networks, Kauffman concludes that they cannot use such chaos, and that things "must settle down into small state cycles—a repertoire of stable behaviors." (Kauffman, 1995, loc. 1058.) He further concludes that basins of attraction of such small cyclic attractors will trap advantageous autocatalytic networks. We could maintain that the reasoning is essentially correct, but that within attractor-basin portraits, chaotic attractors of reasonably mid-dimensional complexity could, and more likely would, serve similar purposes.

There was a quantum jump in the exploration of cellular automata and Boolean networks when several investigations of network properties began to compare continuous (Crutchfield & Young, 1990) and discrete dynamics and computational theory (Langton, 1990; Li et al. 1990: Mitchell, Crutchfield, & Hraber, 1994; Mitchell, Hraber, & Crutchfield, 1993; Packard, 1988). While the lower limit of complexity (fixed point and cyclic) and the upper limit (completely random) can be described with simple algorithms, equations, and computations, in between these extremes, the complexity, when deterministic can be difficult to specify or even to visualize.

So we tend to treat them as probabilistic and summarize their properties with statistical parameters, such as fractal dimensions, Lyapunov exponents, mutual information, correlation lengths, and so on. The salient features of cellular automata can be revealed without recourse to computational theory, even though this has been a focus of interest since their introduction by Ulam and von Neumann (1947).

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[10] References to Kauffman (1995) specify locations in the Kindle edition. This location is between his Figs. 4.1 and 4.2 of Chapter 4. My own preference is to call anything more complex than fixed point and relatively short periodic attractors, "chaos" or "transient chaos", with complete randomness considered as an upper limit of complexity, that is, chaos of infinite dimension.

[11] Once, Bruce Stewart, when he was at Brookhaven National Laboratories, showed me his incredible program for the Iris computer (of *Tron* fame and sadly now an ancient relic) that could launch several trajectories for dynamical systems, allowing one to watch them evolve as the Cartesian space revolved around on the monitor screen. We took a short meal break while five trajectories of a Lorenz system with typical parametric values chased themselves around the screen. Upon returning, the screens seemed blank, but Bruce pointed to two pixels lit up, fixed points at the loci of two of the three saddles that organize the attractor. Bruce referred to these as the result of transitional chaos. Our conjecture here is that all chaos is transitional. Bruce, among other things, is a maestro of the Lorenz system. (Thompson & Stewart, 1986).

3.2. The "Edge of Chaos" and μ—Bifurcations

First, a reminder that the "onset of chaos" (Crutchfield & Young, 1990; Feigenbaum, 1983) is a bifurcation to a chaotic attractor in continuous dynamical systems. Bifurcations occur when a system is unstable, which is maximal at the bifurcation point. Stability occurs where small changes in the vectorfield (such as may be caused by the parameters or ratios of parameters of the equations) make for only small changes in the topology of the system's behaviors, as reflected in their attractor-basin portraits. Stability in discrete networks similarly occurs when minor changes in the tables of logical rules of make for small changes in the evolution of the networks. Instability occurs in the region in parameter space where small parametric changes in the system's vectorfields for continuous dynamical systems, and small parametric changes in the logical rules for discrete networks, make for dramatic topological changes in the attractor-basin portraits (bifurcations).

Many parameters may affect a system's stability and bifurcations in cellular automata. Most of these have centered on such obvious parameters as N, the number of nodes, K, the in-degree of the nodes, and λ[12], the proportion of the table of logical rules for the nodes yielding a given state, e.g. 0 or 1 for the binary case. Langton (1990) explored this parameter using ring lattices. He found bifurcations to chaos at a critical value, λ_c, of the proportion of the rules, λ, and that as λ approached λ_c the duration of the transient phase until the bifurcation became longer. Other measures also showed increasing complexity. Computational complexity was thus low, increased with the approach to λ_c, and then falls off as complexity approaches randomness as λ continues to be increased.

Measures of the complexity of the behavior of networks include the spreading rate of the difference pattern, γ, entropy, S, and mutual information, M (Li et al. 1990). The spreading rate, γ, compares the difference between trajectories within a neighborhood with similar initial segments and evaluates if they become more or less different in the future. Thus, low γ indicates fixed point or cyclic attractors, while higher γ indicates chaotic attractors. Most studies seem to assume the typical route to chaos progresses from fixed point to cyclic to chaotic, and often authors on cellular automata use 'chaos' to mean random or nearly random, and at other times to designate a complex region between lower complexity and random. This complex region between the cyclic and random is sometimes called the 'edge of chaos' and sometimes 'chaos'.

Figure 6 shows γ as a function of λ as λ is systematically varied from rules that all give 0's through the midpoint of rules yielding half 0's and half 1's to those all producing 1's averaged over several simulations.

[12] "λ is not necessarily the best parameter. One can improve on λ as a control parameter in a number of ways. For instance, Gutowitz (1990) has defined a hierarchy of parameterization schemes in which λ is the simplest scheme, mean field theory constitutes the next simplest scheme, and so on. "However, λ suffices to reveal a great deal about the overall structural relationships between the various dynamical regimes in CA rule space, and it is very useful to get a feel for the 'lay of the CA landscape' at this low-resolution level before increasing the resolution and surveying finer details. For one thing, λ helps restrict the area of search to a particularly promising 'spot', which is useful because higher-order parameterizations map CA rule space onto many dimensions, whereas λ is a one-dimensional parameter." Langton, 1990, p. 15.

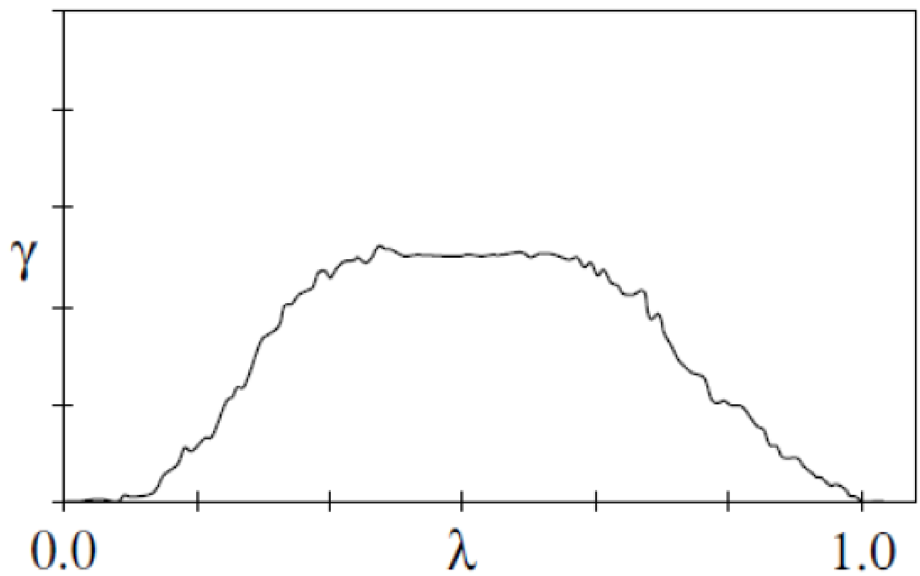


Figure 6. Spreading rates for a large ring network as a function of the proportion λ. It is modified from Packard (1988), as figure 2, in Mitchell et al. (1993). Several replications were run with new Monte Carlo seeds of the initial conditions, each with equal numbers of 0's and 1's, using genetic algorithmically generated changes in the rule tables to adjust λ. Ring lattices with r = 3, k = 2.

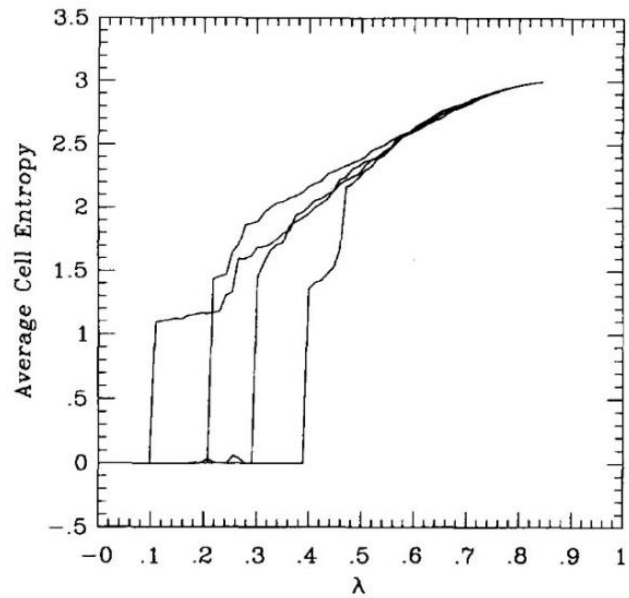
Measures of the Complexity of Network Trajectories (Li et al., 1990)

- *Spreading rate*, γ, describes how trajectories for nodes within a neighborhood become increasingly different over time. These can be averaged over the whole space, or used to define attractive basins, with local and global properties for each attractor.
- *Entropy*, $S = -\Sigma\, p_i \log (p_i)$ also often designated as H
- *Mutual information*, *M*, measures the correlation, that is, similarity of a series in space-time. $M = \Sigma\Sigma p_{ij} \log (p_{ij}/p_i p_j)$

Since several runs are averaged here with differing nodes set to equal numbers of 0's and 1's for the initial condition, and different changes in rule tables for a given λ, the question can be raised as to whether the transitions were gradual, as the curves in figure 4 are, or abrupt.

This question is reminiscent of the debate between all-or-none and gradual learning between Estes' stochastic and Hull's deterministic learning theories of the 1960's (Abraham, 1967; Estes, 1960; Voeks, 1954[13]). Langton showed individual transition events were abrupt events. Figure 7 shows his figures 7 and 8 for entropy, H, rather than the spreading rate, γ, as a function of λ.

[13] Virginia Voeks once told me, that when she did this work for her PhD thesis at Yale under Hull, the gradualist, that she had convinced him that Guthrie's (with whom she did her Masters at the Washington) all-or-none theories were correct, and while he agreed, he claimed it was too late for him to change at that point.



From Langton, 1990.

Figure 7. Superposition of 4 transition events. Note the different λvalues at which the transition take place.

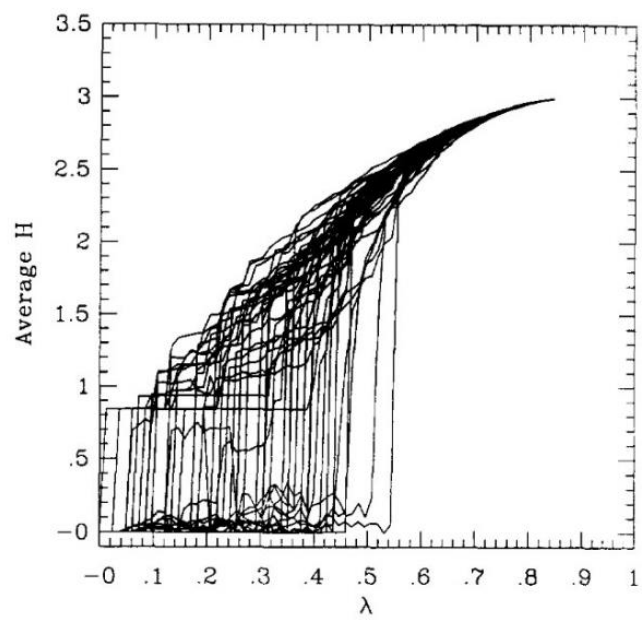


Figure 8. Superposition of 50 transition events showing the internal structure of figure 6.

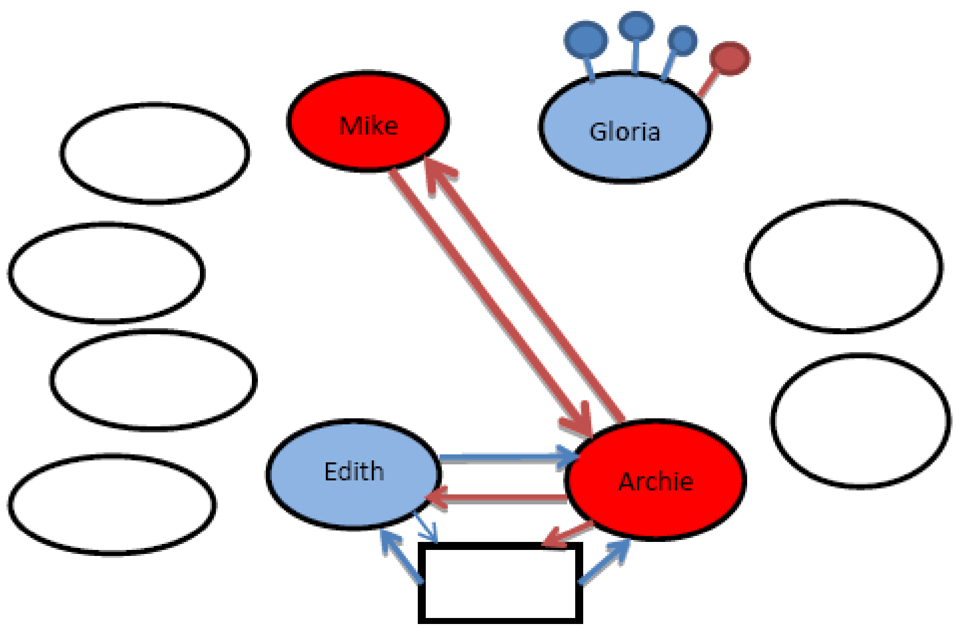
What I want to suggest is that once the larger abrupt bifurcations occur, then as γ increases, the gradual increase in entropy is the result of a rapid sequence, a cascade, of small bifurcations. So we are not in an 'edge of chaos', but in a cascade of chaos-to-chaos bifurcations. Now recall the Ueda explosion (Abraham & Abraham, 2010, figure 15; Figure 8 infra) where tangled homoclinic chaotic attractors bifurcate to heteroclinic tangles among saddles. Each trajectory in a tangle changes from homoclinic to heteroclinic individually within a succession of such changes. These mircro-bifurcations thus occur in cascades, whose complexity remains hidden even from Ueda, just as the details are often left out of network analyses, although many analyses of networks do examine the evolution of complex trees using diagrams of time-series as in figure 3 supra, though much larger and more complex. I like to call these cascades of micro-bifurcations, macro-bifurcational processes, or μ-bifurcations, for both continuous and discrete systems.

Ueda “Duffing’s equation has no statistical parameters and every solution is uniquely determined by the initial condition. The appearance of statistical properties in the physical phenomena in spite of the perfectly deterministic nature of the equation is caused by the existence of noise in the real systems as well as in the global structure of the solutions. A bundle of solutions representing the chaotically transitional process appears in certain domains of the system’s parameters. The details of those stochastic regions have not been discussed as yet.” (Ueda, 1980. P. 137 (425 in the Kyoto repository).	Li, Packard, & Langdon Complex behavior. . . is characterized by long transients and complex space time patterns, [and] by a lack of statistical convergence, for it is not clear that the assumptions needed for computation of statistics hold for this class of rules. When computation of statistics is attempted, entropy is moderate, the spreading rate is roughly zero, and the mutual information is large. (Li, Packard, & Langton, 1990, pp. 5-6.)

4. ALL IN THE FAMILY

Imagine a family with sufficient dysfunction so as to attempt family therapy, and that their network of about 30 members of family and friends over a period of time could have a significant impact on their attitudes and relevance to therapeutic analysis and development. Some of these people come into and out of their lives such that one would have to model it’s evolution as one of preferential attachment and detachment for which I propose, in rough outline, a program that might display their dynamics visually in such a way as to assist family members and therapist to dramatically highlight changes as they occur and thus be a part of the self-organizational nature of the therapeutic process. One could call this a Therapeutic Tracker.

The following network diagram shows nuclear family members, Edith and Archie (mother and father), their daughter, Gloria, and her husband Mike, as four nodes. They are colorized as to their predominant affective modes, positive (blue), and negative (red). This is a directed, weighted graph, and rather than just red vs blue, there would be a rainbow colorization gradient going from red to blue so as to allow averaging into coloring for both the actors (nodes), and their interactions (connections). Only a few nodes and connections are shown here. Let us also assume representation of significant attributes of each actor, that is, within each node, as a continuous or discrete dynamical system. The relative strength of these attributes can wax and wane over time, in self-organizational fashion and are represented as the values of the control parameters of the systems equations should they exist, or by therapeutic rating methods otherwise (see figures 16 & 17 in Beginner's Guide I, Abraham in press). These are shown graphically for Gloria as small satellites. They are not nodes, just visualization of parameters within the system for each node. They could vary not only in magnitude, but new ones might show up during the course of therapy and some might get lopped off. Remaining nodes are various cast members, the rectangle the therapist.



So now, let me suggest some of the options for quantification of the components of the graph. First the size of the nodes can vary over time. These could be varied according to either the total messages arriving, those sent, or both. Second their color could vary either continuously or discretely along the rainbow according to the proportions of positive and negative affective messages, again arriving, departing, or both summed. The connections could vary according to thickness and color, along pretty much the same schemes, but of course would only reflect the pair of nodes that they connect. This scheme allows visual depiction of the evolution of the topology of the network, its system parameters, and four variables (color and size for both nodes and connections reflecting affective quality and quantity of messages) and three types of connections, two directed and one undirected.

In addition to these, if the network contained enough nodes to warrant it, one could run the usual statistical measures of networks, such as various aspects of centrality, clustering, path length, etc. which characterize a network's topography at any moment in time (Barabási, 2002; Easley and Kleinberg, 2010; Abraham, 2013). Given an adequate set of repeated measurements during the evolution of a network, could provide a basis of a continuous dynamical system analysis, including their usual measures such as fractal dimension which could yield additional insights into the family/therapeutic dynamics. For example, strong links between hubs within clusters, if they fail, can break down network integrity, while weak links between nodes that are not hubs may make networks more robust due to providing redundant connections within the network (Reis et al. 2014). Therapeutic communities may be too small to warrant such sophistication, and they could prove counter-productive to the humanistic and experiential aspect of the therapy, maybe useful at times for feedback to individuals or communities, but might gain some traction as research instruments.

Thus there seems to be sufficient motivation to explore network analysis to various traditional psychotherapeutic modalities. Their usefulness could be extended to research, and to having such programs provide a pro-active or participatory role in therapy, such as a patient-run version, e.g. as when used following termination of therapist interventions, and self-help versions.

5. Excursus: The Map Is Not the Territory

Cellular automata, like continuous dynamical systems theory, have limited domains of application for modelling real world processes. Kauffman himself introduces more complex networks before settling down to the development of binary networks in *Universe* (Kauffman, 1995, Figure 3.5, loc. 815). Perhaps the greatest contribution of the brief flare up of the interest in ‘edge-of-chaos’ was to develop the application of such measures as entropy, mutual information, and especially the spreading rate to evolving network trajectories. Von Neumann’s accomplishment was to develop the concept of self-reproducing automata (Von Neumann, 1960; see Mitchell, 2004.) We have pointed to the general concept of cascading micro-bifurcations with gradually changing control parameters, whose generic nature we hope is better captured with the term μ-bifurcation for the macro-view of the process.

Going beyond cellular automata to more flexible network modeling strategies involved first the concept of random networks introduced first by Rapoport (1957) and Erdös and Rényi (1959). Later, the experiments by Milgram (Milgram, 1967; Travers & Milgram, 1969) as analyzed by Watts and Strogatz (1998), showed that many networks displayed characteristics that deviated from random attachment, showing bias and efficiency in communication. Raporport, often overlooked, had also developed analyses of social networks (1957). He was also a champion of conflict resolution and cooperation in evolution.

The next major advance in network theory came from exploration of the World Wide Web by Barabási et al. (Barabási, 2002; Barabási & Albert, 1999). They found that networks grow not by random addition of nodes and connections, but that growth was a log-log function of the popularity existing in prior nodes, a relationship that led to their being called ‘scale free”. The heart of the analysis of such networks comes from a multitude of statistics derived from adjacency matrices of connections among the nodes. These describe the nature of clustering, hubs, network size, and so on, which we do not treat here. Generalizations include pruning off nodes as well as adding them (Saramäki et al. 2013). Also, the networks do not have the rigid structures of cellular automata, and do not require the homogeneity of the agents, and the functions which connect them. They are thus more flexible tools for modelling. The ability to add nodes and variations means that as more variables are seen to interact with the network and thus get added to it, that modelling may come to approach more and more closely the real networks to which they are applied. It thus becomes quickly possible to develop models that surpass the attempt to understand them any more easily than the real world to which they apply (Dutton & Starbuck, 1971). That is we might as well attempt to explain the map from reality.

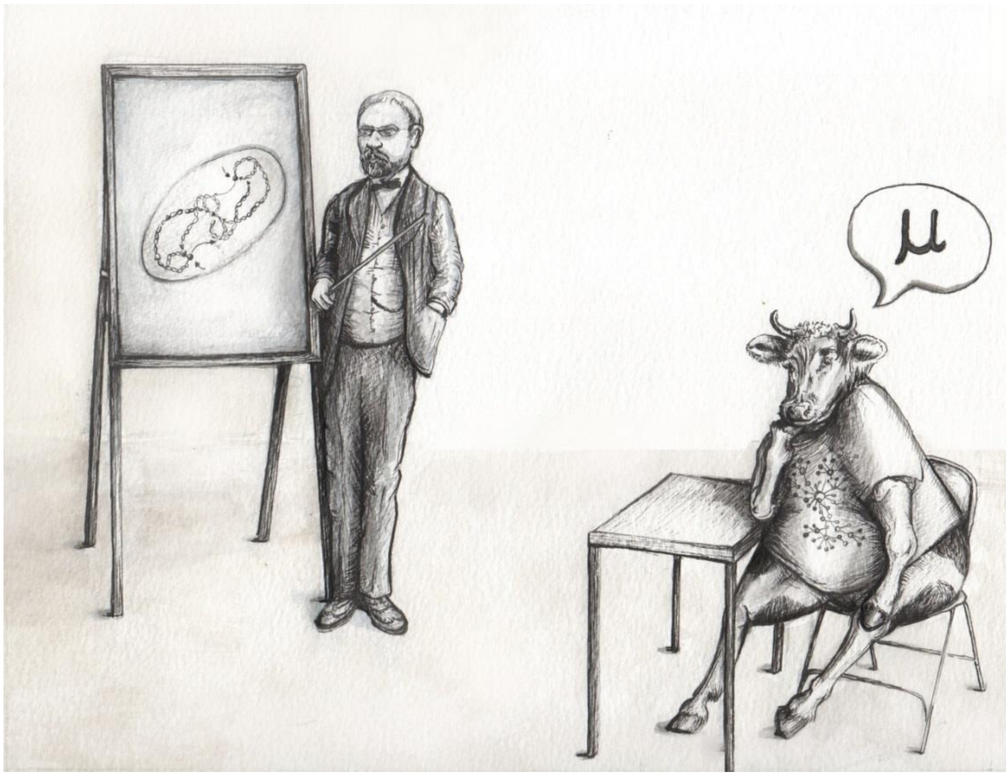
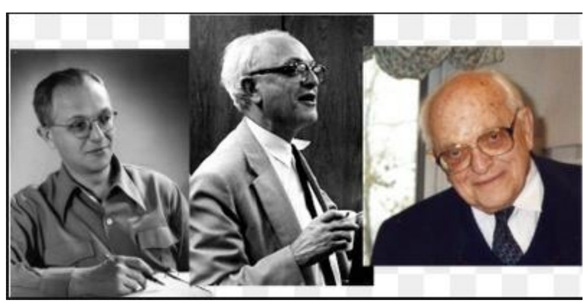


Figure 8. Poincaré explaining the Ueda explosion to a student who was already conceiving the μ-bifurcation. Artwork by grandniece, Laura Abrams.

"The programme of research and action proposed by the systemic approach have a chance to be implemented if science is guided by goals of enlightenment instead [of] appetites of accumulation and power acquisition, as it is, for the most part today. This possible (but by no means guaranteed) byproduct of the information revolution could become an emergency exit from our past predicament." (Rapoport (1996), *The Systemic Approach to Environmental Sociology*, from M Schwaninger, http://www.isss.org/lumrapo.htm)	 Dedication Anatol Rapoport 1911-2007. Polymath, Musician, Mathematical Psychologist, General Systems Theorist, Humanitarian, Philosopher. [From Google image search, 2/1/2014.]

The ‘map is not the territory’ is an old epistemological issue (Bell, 1933; Korzybsky, 1931). We will also not treat this long and fascinating topic here, but merely point out that contemporary complexity theory along with its mathematical and technological tools seem to approach a limit in which the map and the territory become indistinguishable. Paradoxically providing both emergent and submergent properties.

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